# On corner eddies in plane inviscid shear flow

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Local solutions are found for the inviscid shear flow past an acute corner (of included angle $\leqslant \frac{1}{2}\pi$ on the side of the fluid) on an otherwise arbitrary boundary. Unlike irrotational 'corner flows', these solutions are determinate locally, provided that the vorticity is known. Under certain circumstances the existence of a corner eddy may be inferred.

## 1. Introduction

Yih (1959) has shown that a certain inviscid shear flow in a channel (figure 1*a*) includes closed streamlines. The same phenomenon is seen in the shear flow past a semicircular projection on a plane wall (figure 1*b*: details of this solution will be given in § 4).† The purpose of this paper is to point out that such eddies may always be expected under certain conditions, which will be described, and to give simple expressions for the stream function in the immediate neighbourhood of the corner.

We are concerned throughout with plane, inviscid, incompressible shear flow, and mainly with the case of constant vorticity. Then, writing

$$\psi = \int (u\,dy - v\,dx), \quad z = x + iy = r\,e^{i\theta},$$

for the stream function and coordinates, we have

$$\nabla^2\psi = \text{const.} = \omega > 0, \quad \text{say.} \tag{1}$$

We suppose that at some point the tangents to a solid boundary (or to some other streamline) make an angle $\beta \leqslant \frac{1}{2}\pi$, measured on the side of the fluid. Near the corner the fluid is sufficiently stagnant for its motion to be determined by the rotation, so that (for $\omega > 0$) an observer moving with the fluid near the corner keeps the boundary on his *left*. We shall see that the dominant term in the expression for the velocity near the corner is independent of the boundary shape away from the corner, and that this term dominates no matter what irrotational flow past the boundary is added. If now the main flow further out from the corner is such that an observer moving with it keeps the boundary on his *right*, a corner eddy occurs. In this case the local solution near the corner implies the existence of an eddy, but of course the size of the eddy depends on the flow as a whole.

† Despite the apparent similarity of figure 1*b* to figure 1 of Yih (1960) the stream functions in question are quite different.

Flows with constant vorticity are easy to treat analytically, and viscous flows bounded by closed streamlines are known to have constant vorticity in the limit of infinite Reynolds number (Batchelor 1956). Accordingly the behaviour of solutions of (1) near acute corners will be studied in some detail.

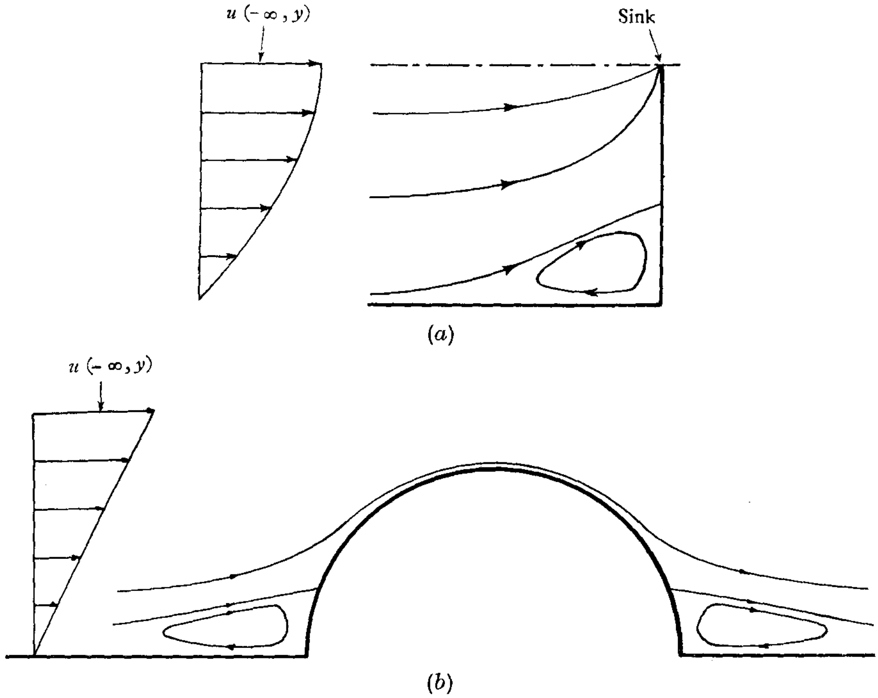



FIGURE 1. (a) Yih's channel flow, (b) shear flow past a semicircular projection.

We also expect our results to hold for the more general inviscid shear flow problem

$$\nabla^2\psi = f(\psi), \tag{2}$$

for if a corner occurs on the streamline $\psi = \psi_0$, such that $f(\psi_0) \neq 0$, it is presumably legitimate to write $f(\psi) \sim f(\psi_0)$ for the purpose of examining the flow very near the corner (where $\nabla\psi = 0$). In the Appendix it is shown that Yih's solution (of an equation of the type (2)) does in fact have the local behaviour predicted by this procedure, but no general proof of the occurrence of our corner flows in solutions of (2) will be given.

Dean (1944) has found flows containing eddies in the lee of a peak in certain solutions of the Stokes slow-motion equations; such eddies are a different species from those considered here.

## 2. Corner flows with constant vorticity

We seek a solution of (1) satisfying $\psi = \text{const.}$ on $\theta = \pm\frac{1}{2}\beta$ ($\beta < \frac{1}{2}\pi$). A particular integral is given by $\psi = \frac{1}{4}\omega r^2$, and the harmonic, complementary function required by the boundary condition may be found by inspection; thus

$$\psi_1 = \tfrac{1}{4}\omega r^2\left(1 - \frac{\cos 2\theta}{\cos\beta}\right) \quad (\beta < \tfrac{1}{2}\pi). \tag{3}$$

The streamlines are the hyperbolae

$$\frac{x^2}{\cos^2(\frac{1}{2}\beta)} - \frac{y^2}{\sin^2(\frac{1}{2}\beta)} = -\frac{8\psi_1 \cos\beta}{\omega \sin^2\beta},$$

and are shown in figure 2*a*: an observer moving with the fluid in the acute angles keeps the boundary on his left, as was stated above. The solution may be interpreted as a corner flow or as a stagnation-point flow; that is, either *AOB* or *AOC*

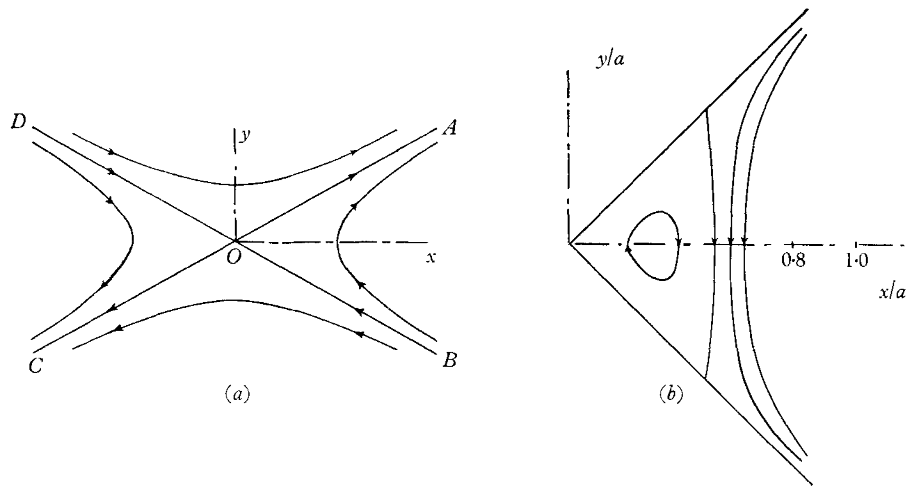



FIGURE 2. Rotational corner flows. (*a*) $\beta < \frac{1}{2}\pi$; (*b*) $\beta = \frac{1}{2}\pi$.

could be a solid boundary, or *O* could be a stagnation point in the body of the fluid. The solution (3) is not unique for the region $|\theta| \leqslant \frac{1}{2}\beta$, since conditions have not been specified on a closed boundary, and an irrotational flow

$$\psi_2 = Cr^{\pi/\beta} \cos\frac{\pi\theta}{\beta}, \tag{4}$$

where *C* is arbitrary, can be added without violating the boundary condition. However, we observe that, since $\pi/\beta > 2$, $\psi_1$ *dominates for sufficiently small values of r*.

The solution (3) becomes infinite for $\beta = \frac{1}{2}\pi$, and we therefore form a linear combination of $\psi_1$ and $\psi_2$ which remains finite as $\beta \to \frac{1}{2}\pi$. Writing

$$\psi_3 = \tfrac{1}{4}\omega\left(r^2 + \lim_{\beta\to\frac{1}{2}\pi} \mathscr{R}\frac{z^{\pi/\beta} - z^2}{\cos\beta}\right),$$

we obtain

$$\psi_3 = \omega\pi^{-1}(\tfrac{1}{4}\pi r^2 + r^2 \log r \cos 2\theta - r^2\theta \sin 2\theta) \quad (\beta = \tfrac{1}{2}\pi).$$

Again we may add an irrotational flow, $\psi_2$, with $\pi/\beta = 2$; with $C = -\omega \log a/\pi$, where *a* is an arbitrary length, we have

$$\psi_4 = \omega\pi^{-1}\{\tfrac{1}{4}\pi r^2 + r^2 \log(r/a) \cos 2\theta - r^2\theta \sin 2\theta\} \quad (\beta = \tfrac{1}{2}\pi). \tag{5}$$

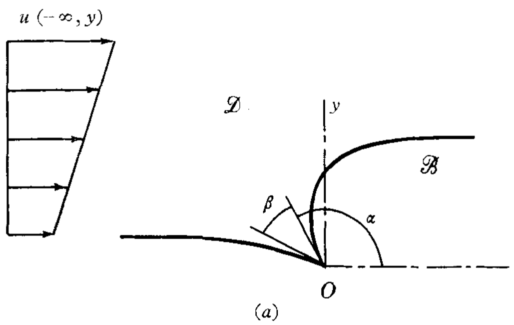
The $r^2 \log r$ term dominates for sufficiently small values of *r*.

The streamlines are shown in figure 2*b*: they are not drawn for $|\theta| > \frac{1}{4}\pi$ because $r^{-1}\partial\psi_3/\partial\theta$ is discontinuous across the negative real axis (if we choose $|\theta| < \pi$), so that the flow is not a real one there.

Now let $\psi$ be a solution of (1) for any field with a streamline which makes an acute angle at some point $O$, and take coordinates with origin at $O$ and with the positive $x$-axis bisecting the angle. The foregoing considerations make it plausible that $\psi_1$ or the logarithmic part of $\psi_3$ provide the dominant term (apart from a constant) in the expansion of $\psi$ about 0, in powers and logarithms of $r$. In the next section we shall prove this assertion.

## 3. The flow near acute corners on arbitrary boundaries

Consider a fluid domain $\mathscr{D}$ whose boundary $\mathscr{B}$ is a Jordan curve with a corner at $O$ of included angle $\beta \leqslant \frac{1}{2}\pi$. Various types of domain are of interest (figure 3): here we consider only the case (figure 3*a*) in which $\mathscr{D}$ is semi-infinite and simply



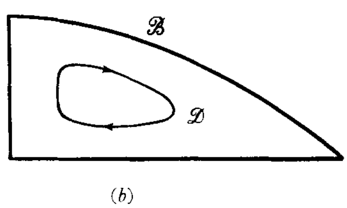




FIGURE 3. Two types of fluid domain to which the present results apply locally.

connected, since for other cases the proof only requires changes in points of detail. The domain can be mapped conformally on to the interior of a circle (Carathéodory 1932; Courant 1950) or, alternatively, on to an upper half-plane. Let

$$z = z(\zeta) = z(\xi + i\eta)$$

be one of the latter transformations, such that the corner $z = 0$ maps on to $\zeta = 0$, and $\mathscr{D}$ on to $\eta > 0$. Let $\mathscr{B}$ be so restricted that we may write, after a suitable choice of constants,

$$\left.\begin{aligned} \text{for } |\zeta| \leqslant \delta, \quad z &= e^{i\alpha}\zeta^b + o(\zeta^b) \quad (0 < b = \beta/\pi \leqslant \tfrac{1}{2}), \\ y(\xi, 0) &= |\xi|^b \sin(\alpha+\beta) + O(|\xi|^c) \quad (\xi \leqslant 0,\ c > b) \\ &= \xi^b \sin\alpha + O(\xi^c) \quad (\xi \geqslant 0). \end{aligned}\right\} \tag{6}$$

A sufficient (but by no means necessary) condition for this is that the two arcs of $\mathscr{B}$ meeting at $O$ have bounded curvature in $-\delta \leqslant \xi \leqslant 0$ and $0 \leqslant \xi \leqslant \delta$, respectively; for then we may first apply the Schwarz–Christoffel mapping appropriate to the tangents at the corner, and then the theorem in § 152 of Carathéodory (1932). We further restrict $\mathscr{B}$ to have $y$ finite and $dy/dx \sim O(x^{-2})$ for $|x| \to \infty$: then for $|\zeta| \to \infty$,

$$z = k_1 \zeta + k_2 \log \zeta + k_3 + O(\zeta^{-1}) \quad (k_1, k_2 \text{ real}).$$

Our problem is to solve (1) with the boundary conditions

$$\left.\begin{array}{c} \psi = \text{const. on } \mathscr{B} \\ \dfrac{\partial \psi}{\partial x} \to 0, \quad \dfrac{\partial \psi}{\partial y} - \omega y - U \to 0, \quad \text{as} \quad |z| \to \infty \text{ in } \mathscr{D}. \end{array}\right\} \tag{7}$$

The solution is

$$\psi = \tfrac{1}{2}\omega y^2 + U k_1 \eta + \psi_I, \tag{8}$$

where $\psi_I$ is a harmonic function required by the boundary condition on $\mathscr{B}$. Representing it by sources on $\eta = 0$, we have

$$\begin{aligned} \psi_I(\xi_1, \eta_1) = \mathscr{I} w_I(\zeta_1) &= \mathscr{I} \frac{\omega}{2\pi} \int_{-\infty}^{\infty} \log(\zeta_1 - \xi) \frac{\partial}{\partial \xi} \{y^2(\xi, 0)\}\, d\xi \\ &= \mathscr{I} \frac{\omega}{2\pi} \left\{ \left( \int_{-\infty}^{\xi_A} + \int_{\xi_B}^{\infty} \right) \log(\zeta_1 - \xi) \frac{\partial}{\partial \xi} \{y^2(\xi, 0)\}\, d\xi \right. \\ &\quad \left. + [\log(\zeta_1 - \xi)\, y^2(\xi, 0)]_{\xi_A}^{\xi_B} + \int_{\xi_A}^{\xi_B} \frac{y^2(\xi, 0)}{\zeta_1 - \xi}\, d\xi \right\}. \end{aligned} \tag{9}$$

Here $A$ and $B$ are points on $\mathscr{B}$ such that $\partial y/\partial \xi$ is continuous for $\xi \leqslant \xi_A$ and $\xi \geqslant \xi_B$ [and $O(\xi^{-2})$ for $|\xi| \to \infty$], and we confine $\zeta_1$ to the domain $\mathscr{D}_1$: $|\zeta_1| \leqslant \epsilon$, $0 \leqslant \arg \zeta_1 \leqslant \pi$, where $\delta > \epsilon > 0$. For $|\xi| > \delta$, the integrands in (9) are continuous functions of $\xi$, uniformly convergent for $|\xi| \to \infty$, and regular functions of $\zeta_1$. Hence, if $F(\zeta_1)$ denotes a function which is regular in $\mathscr{D}_1$,

$$\begin{aligned} w_I(\zeta_1) &= \frac{\omega}{2\pi} \int_{-\delta}^{\delta} \frac{y^2(\xi, 0)}{\zeta_1 - \xi}\, d\xi + F(\zeta_1) \\ &= \frac{\omega}{2\pi} \int_0^{\delta} \left\{ \frac{\lambda^{2b} \sin^2(\alpha + \beta) + O(\lambda^{b+c})}{\zeta_1 + \lambda} + \frac{\lambda^{2b} \sin^2 \alpha + O(\lambda^{b+c})}{\zeta_1 - \lambda} \right\} d\lambda + F(\zeta_1). \end{aligned}$$

Now, by the contour integration associated with the beta function,

$$\begin{aligned} \int_0^{\delta} \frac{\lambda^a}{\zeta \pm \lambda}\, d\lambda &= \mp \frac{\pi \zeta^a}{\sin \pi a} \exp\{i\pi a(-\tfrac{1}{2} \pm \tfrac{1}{2})\} \pm \delta^a \sum_{n=0}^{\infty} \frac{1}{a - n} \left( \mp \frac{\zeta}{\delta} \right)^n \quad (a \neq \text{integer}) \\ &= \zeta \log \zeta - (\zeta \pm \delta) \log(\zeta \pm \delta) \pm \delta \quad (a = 1). \end{aligned}$$

Hence

$$\left| \mathscr{I} \int_0^{\delta} \frac{O(\lambda^{b+c})}{\zeta \pm \lambda}\, d\lambda \right| < M \int_0^{\delta} \frac{\lambda^{b+c} \eta}{(\xi \pm \lambda)^2 + \eta^2}\, d\lambda = O(|\zeta|^{b+c} + |\zeta|) \quad (b + c \neq 1).$$

Omitting the trivial constant $\mathscr{I} F(0)$, we have

$$\begin{aligned} \psi_I(\xi, \eta) &= \mathscr{I} \tfrac{1}{2} \omega \frac{\zeta^{2b}}{\sin 2\beta} \{-\sin^2(\alpha + \beta) + \sin^2 \alpha\, e^{-2i\beta}\} + o(|\zeta|^{2b}) \quad (0 < b < \tfrac{1}{2}) \\ &= \mathscr{I} \frac{\omega}{2\pi} \zeta \log \zeta + O(|\zeta|) \quad (b = \tfrac{1}{2}). \end{aligned}$$

Transforming to $z$, writing $z = r\exp\{i(\tau+\alpha+\frac{1}{2}\beta)\}$, adding the first two terms of (8), and simplifying, we obtain

$$\psi = \tfrac{1}{4}\omega r^2\left(1-\frac{\cos 2\tau}{\cos\beta}\right)+o(r^2) \quad (0<b<\tfrac{1}{2}), \tag{10}$$

$$= \omega\pi^{-1}r^2\log r\cos 2\tau+O(r^2) \quad (b=\tfrac{1}{2}), \tag{11}$$

which is the required result. The sign of $\psi$ yields the local flow directions described in §§ 1 and 2.

## 4. An example

Consider the flow past a semicircular projection, $r = 1$, $0 \leqslant \theta \leqslant \pi$, on a plane wall, $\theta = 0, \pi$, $r \geqslant 1$. The Joukowski transformation

$$\zeta = z+z^{-1} \quad (\zeta=\xi+i\eta)$$

maps the fluid domain on to $\eta > 0$ and the semicircle on to $\eta = 0$, $|\xi| \leqslant 2$. For the differential equation (1) and the boundary conditions (7), we have

$$\psi = \tfrac{1}{2}\omega y^2+U(r-r^{-1})\sin\theta+\psi_I, \tag{12}$$

where $$\psi_I = \mathscr{I}\frac{\omega}{2\pi}\int_{-2}^{2} y^2(\xi^*,0)\frac{1}{\zeta-\xi^*}d\xi^* \quad \text{and} \quad y^2(\xi,0) = 1-\tfrac{1}{4}\xi^2.$$

Hence $$\psi_I = \mathscr{I}\frac{\omega}{2\pi}\left\{\tfrac{1}{4}(\zeta^2-4)\log\frac{\zeta-2}{\zeta+2}+\zeta\right\}$$

$$= \mathscr{I}\frac{\omega}{2\pi}\left\{\frac{1}{2}\left(z-\frac{1}{z}\right)^2\log\frac{z-1}{z+1}+z+\frac{1}{z}\right\}. \tag{13}$$

The case shown in figure 1*b* is for $U = 0$; adding an irrotational flow with $U > 0$ reduces the size of the eddies, and introduces stagnation points on

$$\theta = 0,\pi, \quad 1 < r < \infty.$$

I am indebted to Miss M. Lawson for computing the streamlines of figures 1*b* and 2*b*.

## Appendix: Yih's solution

Yih (1959) has considered the flow in a channel $-1 \leqslant y \leqslant 1$, $x \leqslant 0$, in which there is a sink at the origin and

$$u \to \cos\tfrac{1}{2}\pi y \quad \text{for} \quad x \to -\infty,$$

so that $$\nabla^2\psi = -\tfrac{1}{4}\pi^2\psi.$$

He gives the solution

$$\psi = \frac{2}{\pi}\sin\tfrac{1}{2}\pi y+\frac{4}{\pi^2}\sum_{n=1}^{\infty}A_n(x,y),$$

where $$A_n(x,y) = \frac{1}{n}\left\{1+\frac{(-1)^n}{4n^2-1}\right\}\exp\{(n^2-\tfrac{1}{4})^{\frac{1}{2}}\pi x\}\sin n\pi y.$$

We sum a series whose terms are asymptotically equal to $A_n$. For $n \to \infty$,

$$A_n(x,y) = e^{n\pi x} \sin n\pi y \left\{ \frac{1}{n} - \frac{\pi x}{8}\frac{1}{n^2} + \left(\frac{\pi^2 x^2}{128} + \frac{(-1)^n}{4}\right)\frac{1}{n^3} + O(n^{-4}) \right\}.$$

With $$e^{\pi z} = t, \quad e^{n\pi x} \sin n\pi y = \mathscr{I} t^n,$$

we have $$\sum_{n=1}^{\infty} \frac{(\pm t)^n}{n} = -\log(1 \mp t) \quad (|t| \leqslant 1, t \neq \pm 1),$$

and related series can be 'summed' by integrating both sides from 0 to $t$. Hence

$$\begin{aligned}\mathscr{I}\left[t + \left(1 - \frac{\pi x}{8}\right)\frac{t^2}{2} + \sum_{n=3}^{\infty} t^n \left\{\frac{1}{n} - \frac{\pi x}{8}\frac{1}{n(n-1)} + \left(\frac{\pi x}{8} + \frac{\pi^2 x^2}{128} + \frac{(-1)^n}{4}\right)\frac{1}{n(n-1)(n-2)}\right\}\right] \\ = \mathscr{I}[-\log(1-t) - \tfrac{1}{8}\pi x\{(1-t)\log(1-t) + t\} \\ + (\tfrac{1}{8}\pi x + \tfrac{1}{128}\pi^2 x^2)\{-\tfrac{1}{2}(1-t)^2 \log(1-t) - \tfrac{1}{2}t + \tfrac{3}{4}t^2\} \\ + \tfrac{1}{4}\{-\tfrac{1}{2}(1+t)^2 \log(1+t) + \tfrac{1}{2}t + \tfrac{3}{4}t^2\}] \\ = g(x,y), \quad \text{say.}\end{aligned}$$

The function $\psi - (4/\pi^2) g$ has derivatives with respect to $x$ and $y$ up to the second order, for all $x \leqslant 0$ and all $y$, since the terms of its expansion are $O(t^n n^{-4})$ for $n \to \infty$, with $|t| \leqslant 1$. Hence any singularities of $\psi$ leading to infinite values of $\psi$ or of its first and second partial derivatives must also be present in $(4/\pi^2) g$. Near $x = 0$, $y = -1$, with

$$z + i = z^*, \quad t = e^{\pi z} = -1 - \pi z^* + O(z^{*2}),$$

the local singularity of $\psi$ is therefore given by the $(1+t)^2 \log(1+t)$ term of $g$. Since $z^* = 0$ is a stagnation point, we have

$$\psi = -2/\pi - \mathscr{I} \tfrac{1}{2} z^{*2} \log z^* + O(|z^*|^2).$$

This agrees with (11), for $\omega = \frac{1}{2}\pi$ on the streamline in question.

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